Of Art & Math

Punya Mishra & Gaurav Bhatnagar

Introduction • Symmetry • Self-Similarity • Paradoxes

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Of Art & Math:

Introducing Ambigrams

Punya Mishra
Gaurav Bhatnagar

Mathematicians love puzzles—they love to play with numbers and shapes but often their love can turn to words and other areas that, at least on the surface, have little to do with mathematics. In this article we are going to focus on a very specific kind of artistic wordplay (and its relationship to mathematics) called *ambigrams*. The word ambigram was coined by cognitive scientist Douglas Hofstadter from 'ambi' which suggests *ambiguous* and 'gram' for *letter*. Ambigrams exploit *how* words are written and bring together the mathematics of symmetry, the elegance of typography *and* the psychology of visual perception to create surprising, artistic designs. Most of all, they are great fun!

All right, let's start with the example in Figure 1. Can you read it?

Figure 1. A 180-degree rotation ambigram for the word "Wordplay"

Keywords: *ambigrams, calligraphy, symmetry, perception, palindrome, mapping, transformation, reflection*

Rotating the page you are holding will reveal something interesting. The word stays the same! In other words, it has rotational symmetry.

Thus ambigrams are a way of writing words such that they can be read or interpreted in more than one way. Figure 2 is another one, an ambigram for the word "ambigram."

Figure 2. A 180-degree rotation ambigram for "ambigram"

Incidentally, you may have noticed something interesting in these two examples. In the "wordplay" design each letter of the first half of the word maps onto *one letter* (w to y, o to a, and so on). Some transformations are straightforward (as in the "d" becoming a "p") while others need some level of distortion to work visually (the w-y being the most obvious example). This distortion of course is constrained since whatever shape you come up with has to be readable as specific letters in two different orientations.

Now consider Figure 2, the design for the word "ambigram." There is a lot more distortion going on here. The "stroke" that emerges from the "a" becomes the third leg of the "m." More interesting is how the "m" after the "a" actually maps onto two letters ("r" and "a") when rotated. Isn't it interesting to see that what looks like *one* letter becomes *two* when rotated? On a different note, the g-b transformation is of particular interest to the authors! Can you guess why?

Given that ambigrams work because of the specific mappings of letters (either individually or in groups) to each other implies, that even one change in the letters of the word can lead to a very different design. Thus the solution for the word "ambigrams" (plural) is quite different from the solution for "ambigram" (singular). Note how in Figure 3, many of the mappings have shifted, and the natural "g-b" transformation that made so much sense in the design for "ambigram" has now shifted to a "b-a" transformation while "g" now maps onto itself.

Figure 3.The first of two ambigrams, for "ambigrams." This design reads the same when rotated 180-degrees.

Another important aspect of *why* ambigrams work can be seen in Figure 3. Notice the initial "A" and the final "S." In the case of the "A" the gap at the bottom looks exactly like what it is, a gap. On the other hand, when rotated 180 degrees, our mind imagines a connection across this gap – to make the topmost stroke of the "S." How cool is that!

Rotation is not the only way one can create ambigrams. Figure 4 is another design for the word "ambigrams" –this time as a reflection. This design has bi-lateral symmetry (a symmetry most often found in living things – such as faces, leaves and butterflies). If you place a mirror—perpendicular to the page—in the middle of the 'g', the right half of the design will reflect to become the left part of the word.

Figure 4. Another ambigram for "ambigrams" this time with bilateral symmetry

Designs such as these read the same from right to left. This is a feature of a Palindrome. A palindrome is a word or a sentence that reads the same forwards and backwards. For example, some believe that the first sentence ever spoken was:

Madam, I'm Adam

Notably the response to this palindrome was also a single word palindrome:

Eve

Even longer examples of palindromes can be created. Here is, for instance, a palindromic poem.

> *I think*
>
> *a lot*
>
> *about*
>
> *very little*
>
> *and care*
>
> *very little*
>
> *about*
>
> *a lot*
>
> *I think*

Reverse the sequence of lines (from bottom to top) and you will have the same poem! Here the palindrome is at the level of a line of the poem. The first line is "I think" and so is the last line. Similarly, the second line from both top and bottom is "a lot". The poem is symmetric about the phrase "and care" which comes in the middle of the poem. The symmetry is very similar to the mirror symmetry mentioned earlier, though not quite the same.

Limiting ourselves to just mirror symmetry, we can find many examples of its relevance to mathematics. For example, consider an isosceles triangle, a triangle with two sides equal. It has the same symmetry as the design above.

It is possible to prove that the base angles of an isosceles triangle are equal, just by exploiting this mirror symmetry? Here is a hint:

> *An Isosceles Triangle*
>
> *Looks in the mirror*
>
> *And finds itself un-reversed.*
>
> *"My base angles are equal"*
>
> *it says.*

Visually this can be represented as a triangle-ambigram for the word "isosceles", see Figure 5.

Figure 5. An isosceles triangle that reads "isosceles" when reflected in a mirror

Different types of ambigrams

Every ambigram design need not read the *same* word when rotated and/or reflected. Figure 6 is a design that reads "darpan" (the Hindi word for mirror), and "mirror" (the English word for darpan) when rotated 180 degrees.

Figure 6.The word "darpan" (hindi for mirror) becomes "mirror" on rotation by 180-degrees

So far we have seen ambigrams with a vertical line of symmetry like the designs for "ambigrams" or "isosceles" having a vertical line of symmetry. Hofstadter has called this a "wall reflection." The other is a "lake reflection" such as the example in Figure 7 – where the word "abhikalpa" (the Sanskrit word for architect) which has a horizontal line of symmetry. Mathematically speaking, a wall-reflection is a reflection across the "y-axis" while a "lake-reflection" is a reflection across the "x-axis."

Figure 7. Ambigram for "abhikalpa", an example of a lake reflection

Incidentally, the use of Hindi words in the above two designs brings up an interesting challenge. Is it possible to create an ambigram that can be read in two different languages? Here is the Sanskrit sound “Om” as traditionally written in Devanagri script. This design if rotated 90-degrees magically transforms into the letters “Om” in English!

Figure 8. The Sanskrit word “om”

Figure 9. The English “om” formed by rotating the Sanskrit “om” by 90-degrees.

Not all reflection ambigrams have to be reflected across the x- or y-axes. Consider this design (Figure 10), where the word “right” when reflected across the 45-degree axis reads “angle.” (This design was inspired by a solution first put forth by Bryce Herdt).

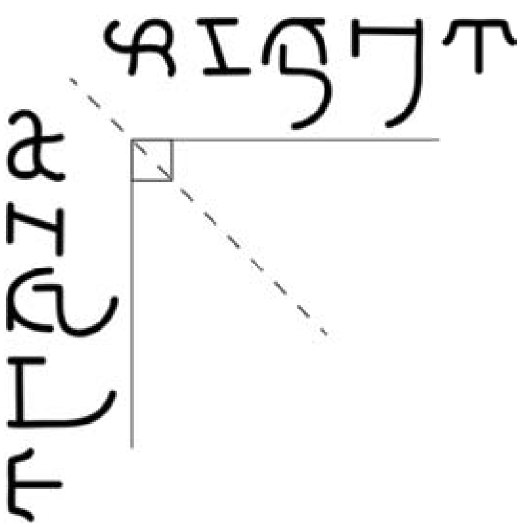


Figure 10. A special “Right angle” made specially for this special magazine

Those who are familiar with tessellations will like the next kind of designs—space-filling ambigrams. See for instance Figure 11, this design for the word “space” – where replications of the word form a network that cover a surface – in this case the surface of a sphere.

Figure 11. A space-filling ambigram for “space”

Here is an example of a rotational chain ambigram for the word “mathematics.” In chain-ambigrams a word is broken into two parts – each of which maps to itself. In Figure 12 “math” maps onto itself and the rest of the word “ematics” maps onto itself.

Figure 12. An ambigram for “mathematics”

Effective chain-ambigrams can be quite rich in meaning. Consider Figure 13. This example of a chain ambigram for “action-re-action” where the letters “-re-“ switch loyalty depending on whether you are reading the top part of the circle or the bottom.

Figure 13. Ambigram for "Action-re-action"

Given this idea of breaking words into shorter ambigramable pieces, it is easy to create such chain-reflection ambigrams as well—such as Figure 14 for the word "reflect." This design will read the same when you hold it up against a mirror (or peer at it from the other side of the page holding it up to a light).

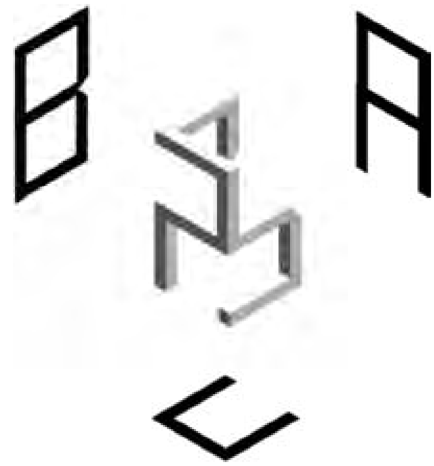


Figure 16. A triplet ambigram for "A","B" and"C"

Even seeing patterns in parts of a word can lead to interesting designs, such as the star-shaped design in Figure 17 for the word Astronomy. In designs like this one takes advantage of specific letters to

Figure 14. A chain-reflection ambigram for "reflect"

A couple of other types of ambigrams are called "figure-ground" ambigrams and "triplets." A figure-ground ambigram is akin to a tessellation – where the space between the letters of a word can be read as another word altogether. What do you see in Figure 15? Good? Evil? Can you see both? Can you see both at the same time? A good pun-ya?

Figure 15. A Figure-Ground ambigram for "Good" and "Evil"

Mathematicians who love solid geometry will love triplets! A triplet is 3-dimensional shape designed in such a way that it casts different shadows depending on where you shine light on it. For instance the design below (Figure 16) is a shape that allows you to see the letters "A," "B" and "C" depending on where you shine light on it.

create visually attractive designs. The designer in this case noted that the letter "R" could be rotated 60-degrees to make the letter "N."

Figure 17. A star shaped ambigram for "astronomy"

Aesthetics, ambigrams & mathematics

Some mathematicians speak of what they do in aesthetic terms. The famous mathematician George Polya remarked: "Beauty in mathematics is seeing the truth without effort." This mirrors

Keat's famous line "Beauty is truth, truth beauty." As Bertrand Russell said, "Mathematics, rightly viewed, possesses not only truth, but supreme beauty." Figure 18 attempts to capture this idea.

Figure 18. A design for "truth & beauty", where Beauty becomes Truth & Truth becomes Beauty.

When mathematicians speak of beauty they usually talk of theorems or proofs that are elegant, surprising, or parsimonious. They speak of "deep" theorems. Mathematical insights that are not obvious, but explained properly seem inevitable. Finally mathematicians delight in doing mathematics, which often means solving problems set by themselves or by other mathematicians.

Effective ambigram designers, in small ways, see the creating of ambigrams as sharing many of these characteristics that mathematicians speak of. The creation of ambigrams can be a highly engaging activity that can lead to seemingly inevitable and yet surprising and elegant solutions. In that sense, both mathematicans and ambigram-artists engage in what we have called "Deep Play" (DP) – a creative, open-ended engagement with ideas through manipulating abstract symbols. We must admit, however, that our teachers have often considered what we do as being more TP (Time Pass) than DP (Deep Play!). We hope we have been able to give you some of the flavor of the art and mathematics of ambigrams. In subsequent articles we will delve deeper into the mathematical aspects of these typographical designs, and use them to communicate mathematical ideas such as symmetry, paradoxes, limits, infinities and much more.

About the authors

PUNYA MISHRA, when not creating ambigrams, is professor of educational technology at Michigan State University. GAURAV BHATNAGAR, when not teaching or doing mathematics, is Senior Vice-President at Educomp Solutions Ltd.

Loving both math and art, Punya's and Gaurav's collaboration began over 30 years ago when they were students in high-school. Since then, they have individually or collectively, subjected their friends, family, classmates, and students to a never ending stream of bad jokes, puns, nonsense verse and other forms of deep play. To their eternal puzzlement, their talents have not always been appreciated by their teachers (or other authority figures). Punya's email address is punya@msu.edu and his website is at http://punyamishra.com. Gaurav's email address is bhatnagarg@gmail.com and his website is at http://gbhatnagar.com

All the ambigrams presented in this article are original designs created by Punya Mishra (unless otherwise specified). Please contact him if you need to use them in your own work.

You should note, dear reader, that Punya and Gaurav have hidden a secret message in this article. If you can work out what it is, or if you have any input, thoughts, comments, original ambigram designs to share, please drop them a note at their e-mail IDs.

Of Art & Math:

Introducing Symmetry

Punya Mishra
Gaurav Bhatnagar

In our November column we introduced the concept of ambigrams—the art of writing words in surprisingly symmetrical ways. Consider an ambigram of the word "Symmetry" (Figure 1).

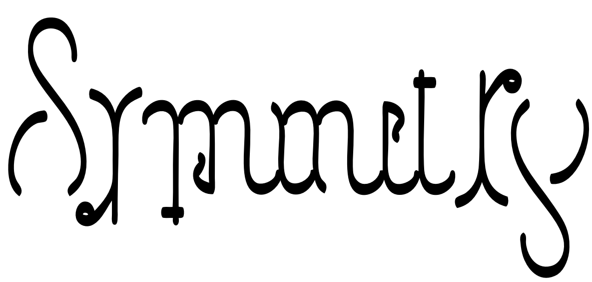


Figure 1. A symmetric ambigram for Symmetry

The design itself displays *rotational symmetry, i.e. it looks the same even when rotated by 180 degrees.* In other words, it remains *invariant* on rotation. Figure 2 shows an ambigram for "invariant" with a similar property.

Keywords: *Ambigrams, calligraphy, symmetry, perception, mapping, transformation, reflection, translation, invariance, function, inverse*

Figure 2. An ambigram for “invariant” that remains invariant on rotation

Invariance can also be seen in reflection. Figure 3 gives a design for the word “algebra” that is invariant upon reflection, but with a twist. You will notice that the left hand side is NOT the same as the right hand side and yet the word is still readable when reflected. So the invariance occurs at the level of meaning even though the design is not visually symmetric!

Figure 3. An ambigram for ‘algebra’ that remains invariant on reflection. But is it really symmetric?

In this column, we use ambigrams to demonstrate (and play with) mathematical ideas relating to symmetry and invariance.

There are two common ways one encounters symmetry in mathematics. The first is related to graphs of equations in the coordinate plane, while the other is related to symmetries of geometrical objects, arising out of the Euclidean idea of congruence. Let’s take each in turn.

Symmetries of a Graph

First let us examine the notions of symmetry related to graphs of equations and functions. All equations in x and y represent a relationship between the two variables, which can be plotted on a plane. A graph of an equation is a set of points (x, y) which satisfy the equation. For example, $x^2 + y^2 = 1$ represents the set of points at a distance 1 from the origin—i.e. it represents a circle.

Let (a, b) be a point in the first quadrant. Notice that the point $(-a, b)$ is the reflection of (a, b) in the y-axis (See Figure 4a). Thus if a curve has the property that $(-x, y)$ lies on the curve whenever (x, y) does, it is symmetric across the y-axis. Such functions are known as even functions, probably because $y = x^2, y = x^4, y = x^6,\ldots$ all have this property. Figure 4b shows the graph of $y = x^2$; this is an even function whose graph is a parabola.

Similarly, a curve is symmetrical across the origin if it has the property that $(-x, -y)$ lies on the curve whenever (x, y) does. Functions whose graph is of this kind are called odd functions, perhaps because $y = x, y = x^3, y = x^5,\ldots$ all have this property. See Figure 4c for an odd function.

A graph can also be symmetric across the x-axis. Here $(x, -y)$ lies on the curve whenever (x, y) does. The graph of the equation $x = y^2$ (another parabola) is an example of such a graph. Can a (real) function be symmetric across the x-axis?

Figure 5 shows a chain ambigram for “parabola”. Compare the shape of this ambigram with the graph in Figure 4b. The chain extends indefinitely, just like the graph of the underlying equation!

Figure 5. A parabolic chain ambigram for “parabola”

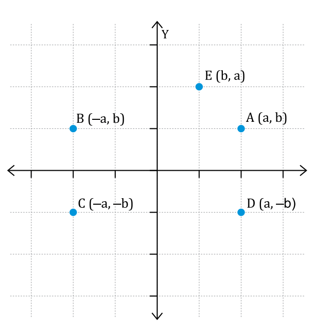


Figure 4a. The point *A (a, b)* and some symmetric points

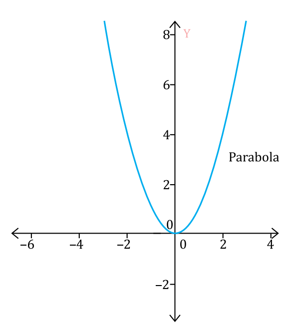


Figure 4b. An even function: $y = x^2$

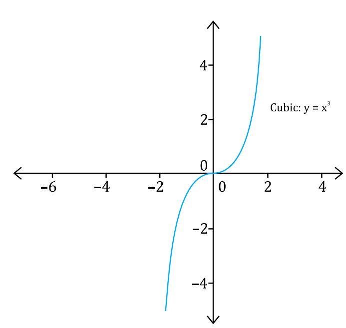


Figure 4c. An odd function: $y = x^3$

The ambigram for "axis of symmetry" (Figure 6) is symmetric across the y-axis. You can see a red y as a part of the x in the middle. So this ambigram displays symmetry across the y axis. At the same time it is symmetric across the letter x!

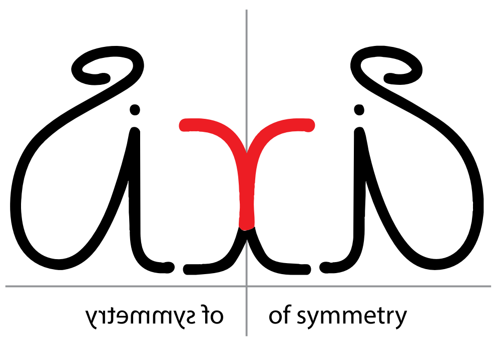


Figure 6. The axis of symmetry: Is it the y-axis or the x?

Another possibility is to interchange the x and y in an equation. Suppose the original curve is C_1 and the one with x and y interchanged is C_2. Thus if (x, y) is a point on C_1, then (y, x) lies on C_2. By looking at Figure 4a, convince yourself that the point (b, a) is the reflection of (a, b) in the line $y = x$, the straight line passing through the origin, and inclined at an angle of 45° to the positive side of the x-axis. Thus the curve C_2 is obtained from C_1 by reflection across the line $y = x$. If C_1 and C_2 (as above) are both graphs of functions, then they are called inverse functions. An example of such a pair: the *exp* (exponential $y = e^x$) and *log* (logarithmic y = ln x) functions (Figure 7).

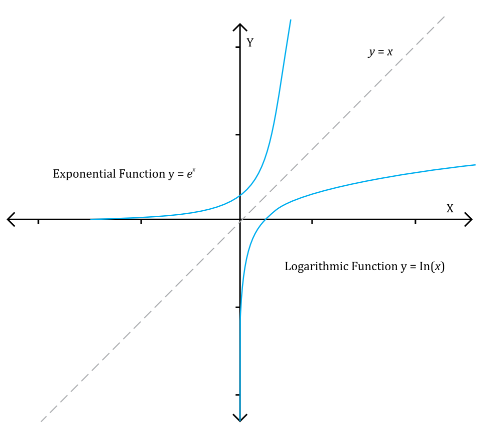


Figure 7. Inverse functions are symmetric across the line $y=x$.

Figure 8 is a remarkable design that where *exp* becomes *log* when reflected in the line $y = x$.

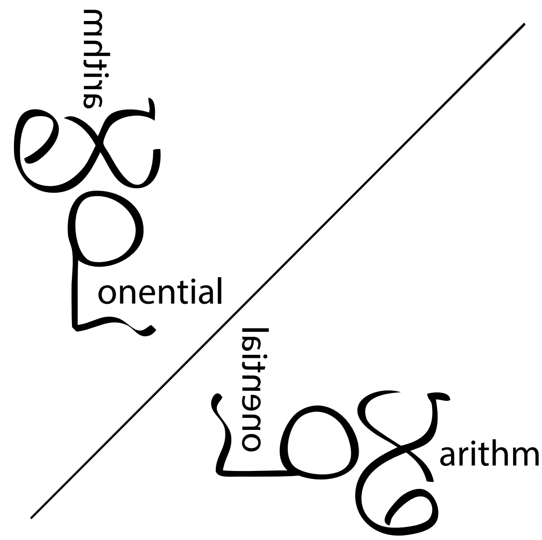


Figure 8: *Exp* becomes *Log* when reflected across the diagonal line!

A great example of an inverse function is the hyperbola $y = 1 / x$, defined for all non-zero real numbers x (Figure 9). Its inverse is obtained by interchanging x with y. But $x = 1 / y$ can be written $y = 1 / x$. So it is its own inverse, and thus symmetric across the line $y = x$.

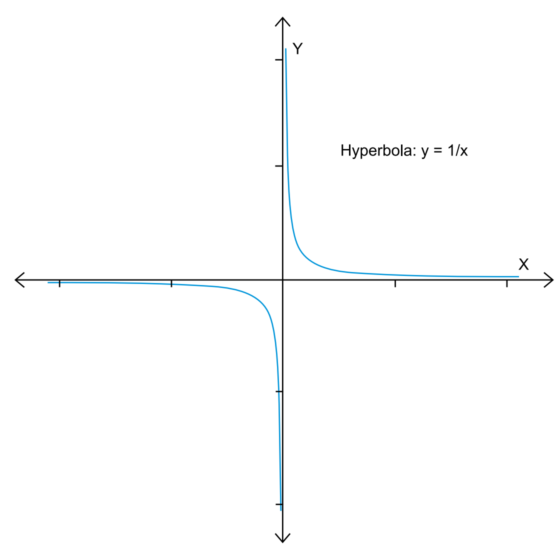


Figure 9. The symmetrical graph of the hyperbola. It is its own inverse. And it's odd, too!

The ambigram for "inverse" in Figure 10 is inspired by the hyperbola.

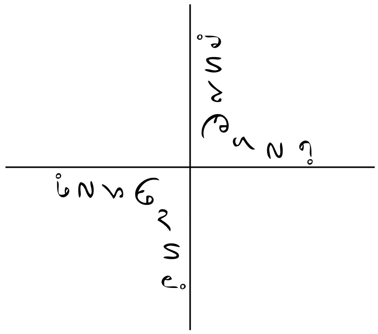


Figure 10. An ambigram for the word "inverse" shaped like a hyperbola.

It is symmetric across the origin and across the line joining the two S's.

Seeking congruence

Another type of symmetry consideration arises from the notion of congruence in plane geometry. Two objects are considered to be congruent if one object can be superposed on the other through rotation, reflection and/or translation.

Figure 11 shows an ambigram of the word "rotate".

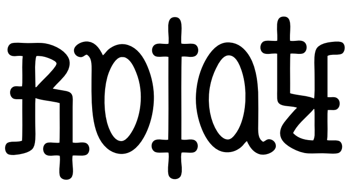


Figure 11. An ambigram for "rotate". What happens when you rotate it through 180°?

This leads to the question: if we can rotate "rotate" can we reflect "reflect"? Figure 12 is an ambigram for "reflect" that is symmetric around the vertical line in the middle.

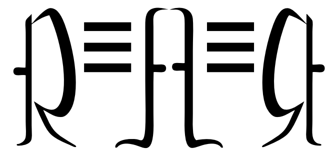


Figure 12. An ambigram for "reflect." What happens when you hold it to a mirror?

Finally, the third operation is translation. An example of this symmetry is shown by the chain ambigram for sine in Figure 13.

Figure 13. A sine wave ambigram. It displays translation symmetry.

The sine function satisfies many symmetry properties. Perhaps the most important of them is that it is periodic, i.e., if you shift (in other words, translate) the functions by 2π, then you get the same function back. In addition, it is an odd function, and the ambigram is both periodic and odd.

One can of course combine these transformations. This is best understood by looking at the symmetries of an equilateral triangle (see Figure 14) involving both rotation and reflection.

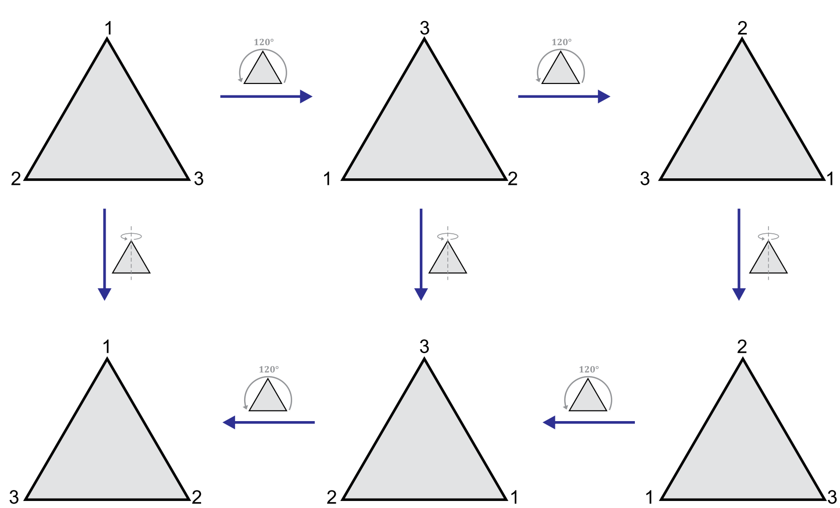


Figure 14. The 6 symmetries of an equilateral triangle.

The 6 symmetries of the equilateral triangle are all found from two fundamental operations: (a) rotation by 120°; and (b) reflection across the line passing through the vertex 1, and At Right Angles to the base of the triangle. Figure 15 shows ambigrams for the word triangle and pentagon. Do they display all the symmetries of the equilateral triangle and regular pentagon (respectively)?

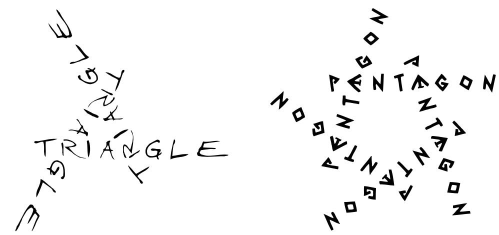


Figure 15: Ambigrams for "triangle" and "pentagon" showing rotational symmetries.

In conclusion

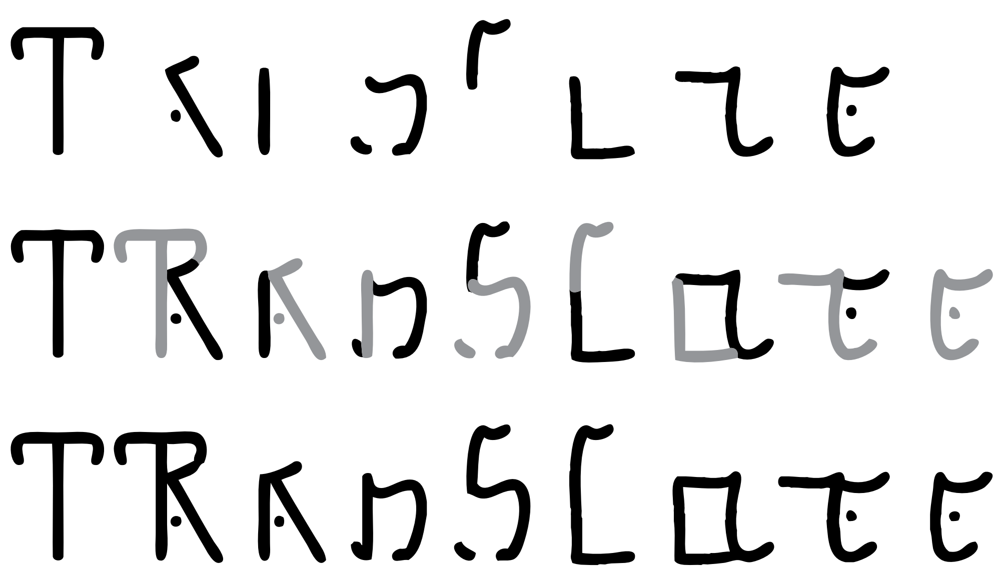
Clearly we have just scratched the surface of the power of symmetry as an idea in mathematics. The philosopher Aristotle once observed that, "the mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful." We agree with Aristotle, but perhaps we would have said "arts" instead of "sciences." In our next article, we will continue to use ambigrams to explore more beautiful mathematical ideas.

Our last article had a secret message. The first letter of each paragraph read: "Martin Gardner lives on in the games we play". This is our homage to Martin Gardner whose writings inspired us when we were growing up. This article has a different puzzle (see Figure 16).

Figure 16. Can you translate this hieroglyphic code? The answer appears below.

ANSWER TO THE PUZZLE

We asked you to "translate" the code. Once you translate (i.e. move) the shapes and align them, you get the answer—the word "Translate." An example of translation symmetry!



About the authors

PUNYA MISHRA, when not creating ambigrams, is professor of educational technology at Michigan State University. GAURAV BHATNAGAR, when not teaching or doing mathematics, is Senior Vice-President at Educomp Solutions Ltd.

Loving both math and art, Punya's and Gaurav's collaboration began over 30 years ago when they were students in high-school. Since then, they have individually or collectively, subjected their friends, family, classmates, and students to a never ending stream of bad jokes, puns, nonsense verse and other forms of deep play. To their eternal puzzlement, their talents have not always been appreciated by their teachers (or other authority figures). Punya's email address is punya@msu.edu and his website is at http://punyamishra.com. Gaurav's email address is bhatnagarg@gmail.com and his website is at http://gbhatnagar.com

All the ambigrams presented in this article are original designs created by Punya Mishra (unless otherwise specified). Please contact him if you need to use them in your own work.

You, dear reader, are invited to share your thoughts, comments, math poems, or original ambigrams at the addresses above.

A RIVER PUZZLE FROM MASTER PUZZLIST SAM LOYD

Two ferry boats ply the same route between ports on opposite sides of a river. They set out simultaneously from opposite ports, but one is faster than the other, so they meet at a point 720 yards from the nearest shore. When each boat reaches its destination, it waits 10 minutes to change passengers, then begins its return trip. Now the boats meet at a point 400 yards from the other shore. How wide is the river?

Comment from Sam Loyd: "The problem shows how the average person, who follows the cut-and-dried rules of mathematics, will be puzzled by a simple problem that requires only a slight knowledge of elementary arithmetic. It can be explained to a child, yet I hazard the opinion that ninety-nine out of every hundred of our shrewdest businessmen would fail to solve it in a week. So much for learning mathematics by rule instead of common sense which teaches the reason why!"

See ***page* 53** for the solution.

Source: http://www.futilitycloset.com/category/puzzles/

in the classroom

Of Art and Math

Self-Similarity

You have wakened not out of sleep, but into a prior dream, and that dream lies within another, and so on, to infinity, which is the number of grains of sand. The path that you are to take is endless, and you will die before you have truly awakened
— Jorge Luis Borges

PUNYA MISHRA
GAURAV BHATNAGAR

The Argentinian author, Jorge Luis Borges, often wrote about his fear of *infinity*—the idea that space and/or time could continue forever. Though Borges' response may appear somewhat overblown, who amongst us has not felt a frisson of excitement when thinking of the infinite and our relative insignificance in front of it. Borges' quote of reality being a dream within a dream within a dream ad infinitum reminds us of the hall of mirrors effect—the seemingly infinite reflections one generates when one places two mirrors in front of each other—the same object over and over and over again.

This idea of infinite reflections can be seen in the chain ambigram in Figure 1 for the word *reflect*. In this design the "RE" and "FLECT" are written in a mirror-symmetric manner, which means that if we repeat this design over and over again it will read the same when held up against a mirror. (For a different ambigram for reflect, see *Introducing Symmetry,* in the March 2014 issue of *At Right Angles).*

Figure 1. An ambigram of "reflect," reflecting the infinite reflections in a pair of mirrors

This idea of repeating a similar shape (often at a different scale) over and over again, is called *self-similarity*. In other words, a self-similar image contains copies of itself at smaller scales. A simple example appears in Figure 2: a repeated pattern for a square that is copied, rotated and shrunk by a factor of $1/\sqrt{2}$.

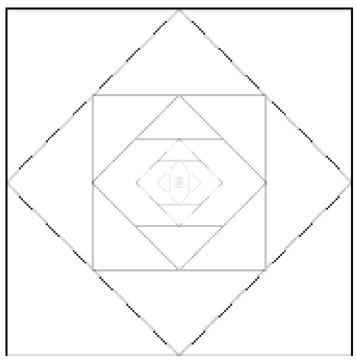


Figure 2. A self–similar design

Of course you can do this with typographical designs as well, such as the design for the word "Zoom" in Figure 3.

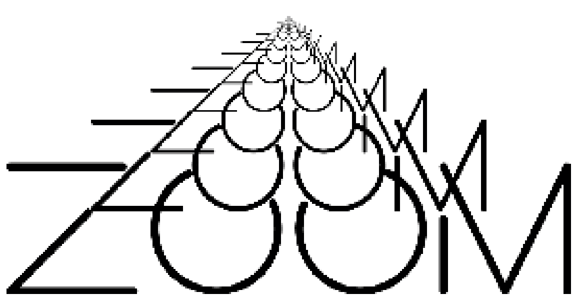


Figure 3. A self-similar ambigram for ZOOM

Examining self-similarity leads to a discussion of infinity, iteration and recursion, some of the ideas we discuss in this article.

Before taking a serious look at self-similarity, we present (see Figure 4) a rotational-ambigram of "self-similarity," which is *not* self-similar. However, below it is another version of the same design, where the word "self" is made up of little rotationally symmetric pieces of "self" and similarity is made up of little ambigrams of "similarity" and, most importantly the hyphen between the words is the complete ambigram for "self-similarity." So this leads to the question: What do you think the hyphen in the hyphen is made of?

Self similarity and Fractals

Self-similar shapes are all around us, from clouds to roots, from branches on trees to coastlines, from river deltas to mountains. The idea of self-similarity was popularized by Benoit B. Mandelbrot, whose 1982 book "The Fractal Geometry of Nature" showed how self-similar objects known as 'fractals' can be used to model 'rough' surfaces such as mountains and coast-lines. Mandelbrot used examples such as these to explain how when you measure a coastline the length of the line would increase as you reduced the unit of measurement. Such convoluted folds upon folds that lead to increased length (or in the case of 3-d objects, increased surface area) can be seen in the structure of the alveoli in the human lungs as well as in the inside of our intestines. The volume does not increase by much, while the surface area increases without limit.

Figure 5 is an ambigram of "Fractal" which illustrates Mandelbrot's own definition of fractals: *A fractal denotes a geometric shape that breaks into parts, each a small scale model of the original*.

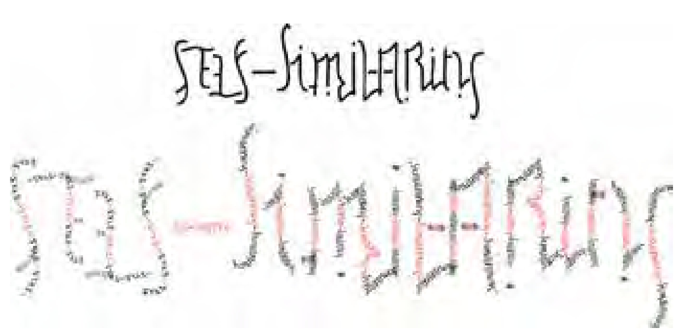


Figure 4: (Top) A rotational ambigram for "self-similarity." (Bottom) The strokes in the first ambigram are now replaced by words. The "self" is made up of tiny versions of "self" and "similarity" of smaller versions of "similarity" (each of which are ambigrams of course). That is not all, the hyphen is made up of a tiny version of the entire design!

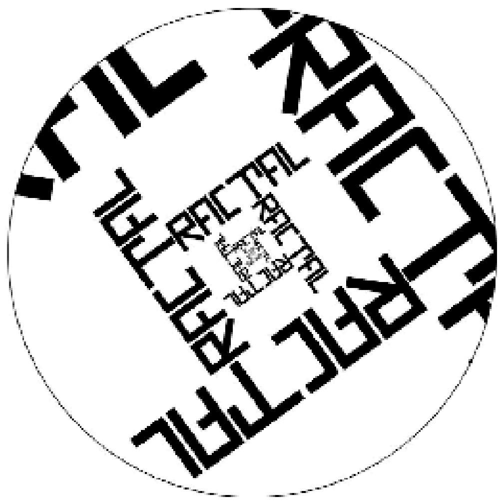


Figure 5: A self-similar, fractal ambigram for "Fractal"

In other words, fractals are geometrical shapes that exhibit invariance under scaling i.e. a piece of the whole, if enlarged, has the same geometrical features as the entire object itself. The design of Figure 6 is an artistic rendition of a fractal-like structure for the word "Mandelbrot".

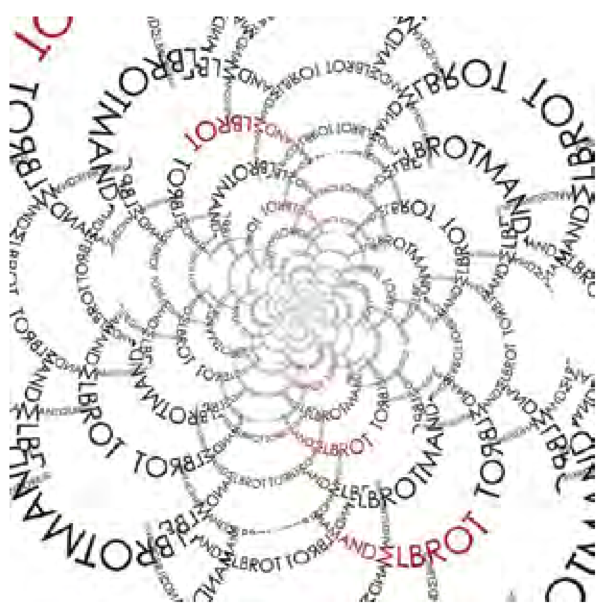


Figure 6. A fractal ambigram for "Mandelbrot"

Speaking of Mandelbrot, what *does* the middle initial "B" in Benoit B. Mandelbrot stand for? A clue is provided in Figure 7.

Figure 7. Puzzle: What does the B in "Benoit B Mandelbrot" stand for?
Answer at the end of the article.

It is clear that the idea of infinity and infinite processes are an important aspect of fractals and self-similarity. We now examine the concept of infinity typographically and mathematically.

Infinity

Infinity means *without end*, or limitless. Mathematically speaking, a finite set has a definite number of elements. An infinite set is a set that is not finite. The word infinity is also used for describing a quantity that grows bigger and bigger, without limit, or a process which does not stop.

Figure 8 has two designs for "infinity" subtly different from each other. Notice how in the first design the chain is created by "in" mapping to itself and "finity" mapping to itself. In contrast the second design breaks the word up differently, mapping "ity" to "in" and "fin" to itself.

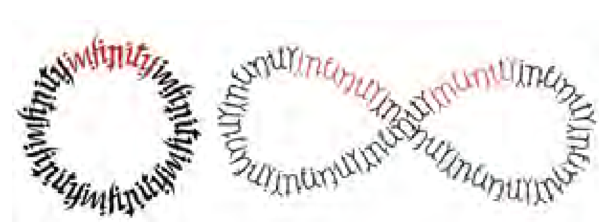


Figure 8. Two ambigrams for "infinity".
The first wraps around a circle and the second says infinity by word and symbol!

The first design wraps “infinity” around a circle. You can go round and round in a circle, and keep going on, so a circle can be said to represent an infinite path but in a finite and understandable manner. The second design is shaped like the symbol for infinity!

In keeping with the idea of self-similarity here are two other designs of the word “infinite”. In fact there is a deeper play on the word as it emphasizes the *finite* that is *in* the *infinite.* The two designs in Figure 9 capture slightly different aspects of the design. The first focuses on mapping the design onto a sphere while the second is a self-similar shape that can be interpreted in two different ways. Either being made of an infinite repetition of the word “finite” *or* the infinite repetition of the word “infinite” (where the shape that reads as the last “e” in the word “finite” can be read as “in” in the word “infinite” when rotated by 90 degrees).

Infinities are difficult to grasp and when we try to apply the rules that worked with finite quantities things often go wrong. For instance, in an infinite set, *a part of the set can be equal to the whole*! The simplest example is the set of natural numbers, and its subset, the set of even numbers.

The set $\mathbb{N} = \{1, 2, 3, 4, ...\}$ of natural numbers is infinite. Now consider the set of even numbers $\mathbb{E} = \{2, 4, 6, 8, ...\}$ Clearly, the set of even numbers has half the number of elements of the set of natural numbers, doesn’t it?

But not so quick! Things are tricky when it comes to infinite sets. We need to understand what it means for two sets to have an equal number of elements. Two sets have an equal number of elements when they can be put in one to one correspondence with each other. Think of children sitting on chairs. If each child can find a chair to sit on, and no chair is left over, then we know that each child *corresponds to* a chair, and the number of children is the same as the number of chairs.

Returning to the natural numbers, each number n in $\mathbb{N}$ corresponds to the number $2n$ in $\mathbb{E}$. So every element of $\mathbb{N}$ corresponds to an element of $\mathbb{E}$ and vice versa.

Thus though one set may intuitively look like it is half the other it is in fact not so! Our intuition is wrong, the sets $\mathbb{E}$ and N have the same number of elements. Since $\mathbb{E}$ is a part of $\mathbb{N}$, you can see that when it comes to infinite sets, *a part can be equal*

Figure 9. Two ambigrams for “infinite”, a play on the finite in infinite. Is the second design an infinite repetition of the word “finite” or “infinite?”

to the whole. In fact, this part-whole equivalence has sometimes been used to define an infinite set.

Another interesting example where a part is equal to the whole, is provided by a fractal known as the Sierpinski Carpet.

The Sierpinski Carpet

The Sierpinski Carpet, like all fractals, is generated using the process of iteration. We begin with a simple rule and apply it over and over again.

Begin with a unit square, and divide the square into 9 equal parts. Remove the middle square. Now for each of the remaining 8 squares, we do the same thing. Break it into 9 equal parts and remove the middle square. Keep going on in this way till you get this infinitely filigreed Swiss-cheese effect. See Figure 10 for the first couple of steps and then the fifth stage of the carpet.

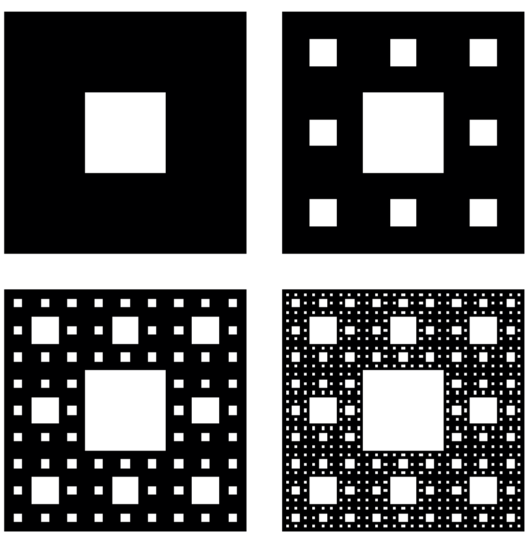


Figure 10. The Sierpinski Carpet

Which leads to the question: What is the total area of all the holes? Here is one way of computing the area of the holes in the Sierpinski Carpet. The first hole has area 1/9. In Step 2, you will remove 8 holes, each with area 1/9 th of the smaller square; so you will remove 8 holes with area $1/9^2$ or $8 \times \frac{1}{9^2} = 8/9^2$. In Step 3, for each of the smaller 8 holes, we remove 8 further holes with area $1/9^3$, so the area removed is $8^2/9^3$. In this manner it is easy to see that the total area of the hole is:

$$\frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3} + \frac{8^3}{9^4} + \cdots = \frac{1}{9}\left(1 + \frac{8}{9} + \frac{8^2}{9^2} + \frac{8^3}{9^3} + \cdots\right)$$

$$= \frac{1}{9} \times \frac{1}{1 - {}^{8}\!/_{9}} = 1.$$

To see why, we use the formula for the sum of the infinite Geometric Series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \text{ for } -1 < x < 1.$$

How crazy is that! The area of the holes (taking away just 1/9th of a square at a time) is equal to the area of the unit square! *Thus the hole is equal to the whole!*

This seemingly contradictory statement has inspired the following design—where the words *whole* and *hole* are mapped onto a square – with the letter o representing the *hole* in the Sierpinski carpet. Of course as you zoom in, the *whole* and *hole* keep interchanging. We call this design *(w)hole in One* (in keeping with the idea the area of the *hole* is equal to the *whole* of the unit square).

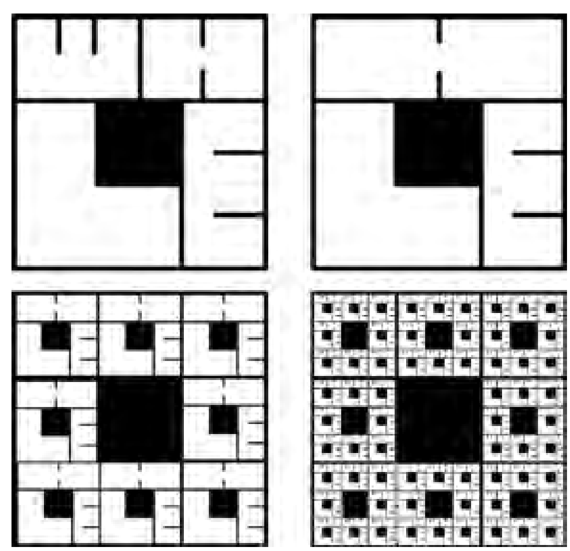


Figure 11. Fractal Ambigrams for "WHOLE" and "HOLE", a (w)hole in One.

The repetitive process of applying a set of simple rules that leads amazing designs like the Sierpinski carpet (and other fractal shapes) is called iteration.

Graphical interpretations of iteration

The process of iteration can be used to generate self-similar shapes. Graphically, we simply superimpose the original shape with a suitably scaled down version of the initial shape, and then repeat the process. The nested squares of Figure 2 is perhaps the simplest example of creating a self-similar structure using this process.

Essentially, such figures emerge from the repeated application of a series of simple steps—a program as it were, applied iteratively to the result of the previously applied rule. In this manner we can arrive at shapes and objects that are visually rich and complex.

Here is another, more creative, way to graphically interpret the idea of iteration.

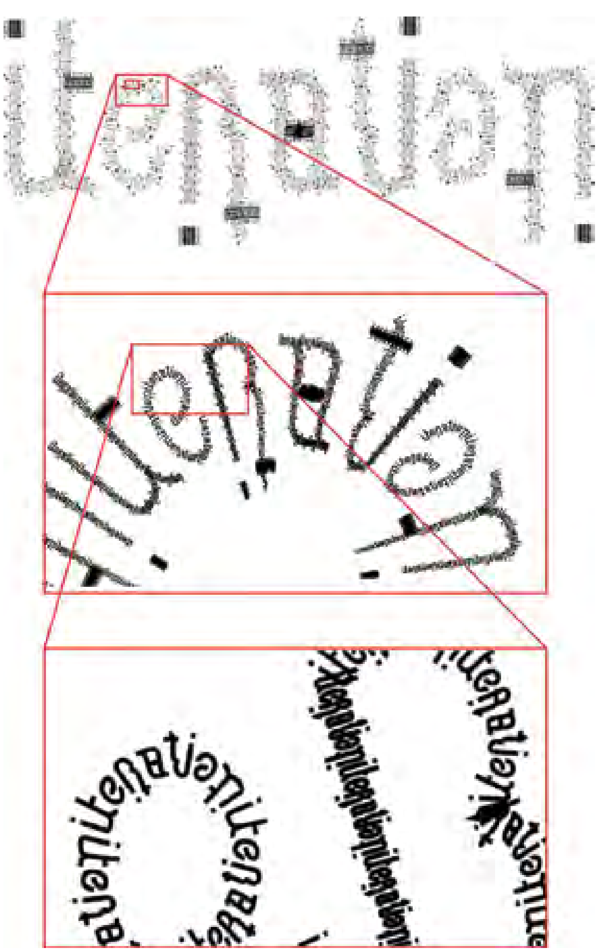


Figure 12: An ambigram of "iteration", illustrating a graphical approach to a part can be equal to the whole.

At one level the first ambigram in Figure 12 can be read as a rotational ambigram for the word "iteration." However if you zoom into the design (see zoomed figures below) you will see that each of the strokes is made of smaller strokes that in turn spell *iteration*.

In fact you can go down one more level and see "iteration" all over again. Theoretically we could do this forever, (within the limits of computational technology and visual resolution of screen, print and eye!). A similar idea is explored in the design of the word "self-similarity" (Figure 4) specifically in the design of the hyphen.

There are other fascinating examples of such iterative techniques, one of which we examine next.

The Golden Mean

Another example of a mathematically and visually interesting structure is the Golden Rectangle (and its close relative the Golden Mean). The Golden Mean appears as the ratio of the sides of a Golden Rectangle. A Golden Rectangle is such that if you take out the largest square from it, the sides of the resulting rectangle are in the same ratio as the original rectangle. Suppose the sides of the Golden Rectangle are a and b, where b is smaller than a. The ratio a/b turns out to be the Golden Mean (denoted by ϕ). The largest square will be of side b. Once you remove it, the sides of the resulting rectangle are b and $a - b$. From this, it is easy to calculate the ratio a/b and find that it equals .

$$\phi \ = \frac{1+\sqrt{5}}{2} = 1.618033988 \ldots.$$

If you begin with a Golden Rectangle and keep removing the squares, you will get a nested series of Golden Rectangles (see the underlying rectangles in Figure 14). The resulting figure shows self-similarity.

You may connect the diagonals using a spiral to obtain an approximation to what is called the Golden Spiral. Figure 13 shows an ambigram of "Golden Mean", placed in the form of a Golden Spiral inside a series of nested Golden Rectangles.

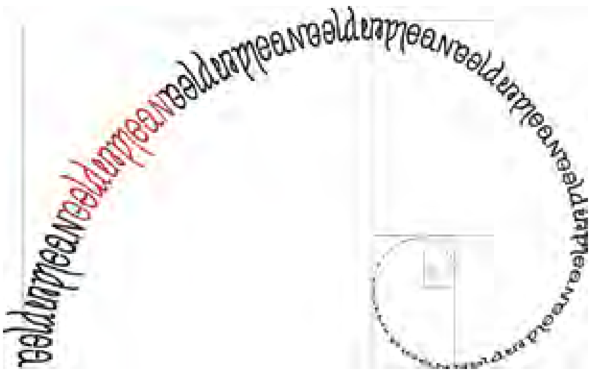


Figure 13. A rotationally symmetric chain-ambigram for the phrase "Golden Mean" mapped onto a Golden Spiral.

The Golden Mean appears in different contexts, in mathematics, in artistic circles, and even in the real world. It is closely related to the Fibonacci Numbers, namely 1, 1, 2, 3, 5, 8, 13, 21, ... Note that the Fibonacci Numbers begin with 1 and 1, and then each number the sum of the previous two numbers. If you take the ratio of successive Fibonacci numbers, the ratio converges to the Golden Mean.

The Fibonacci numbers are an example of a recursively defined sequence, where a few initial terms are defined, and then the sequence is built up by using the definition of the previous term (or terms).

Recursion and Pascal's Triangle

Recursion is similar to iteration. While iteration involves applying a simple rule to an object repeatedly, like in the creation of the Sierpinski Carpet, recursion involves using the results of a previous calculation in finding the next value, as in the definition of the Fibonacci numbers.

Fractals are usually obtained by iteration. Thus it is rather surprising that the fractal of Figure 14, called the Sierpinski triangle, may also be obtained using a recursive process.

The triangle in Figure 14 is a binary Pascal's triangle, where you use binary arithmetic (where 0 + 0 = 0; 0 + 1 = 1 and, 1 + 1 = 0) to create the Pascal's triangle. The recursion is as follows: Each row and column begins and ends with a 1. Every other number is found by the (binary) addition of numbers above it. The formula for the recursion is

$$F(n+1, k) = F(n, k-1) + F(n, k)$$

where $F(n,k)$ is the term in the nth row and kth column, for $n = 0, 1, 2, 3, \ldots$ and $k = 0, 1, 2, 3, \ldots$ and the rules of binary arithmetic are used. In addition, we need the following values:

$$F(n, 0) = 1 = F(n,n).$$

This recurrence relation is the recurrence for generating Pascal's Triangle, satisfied by the Binomial coefficients.

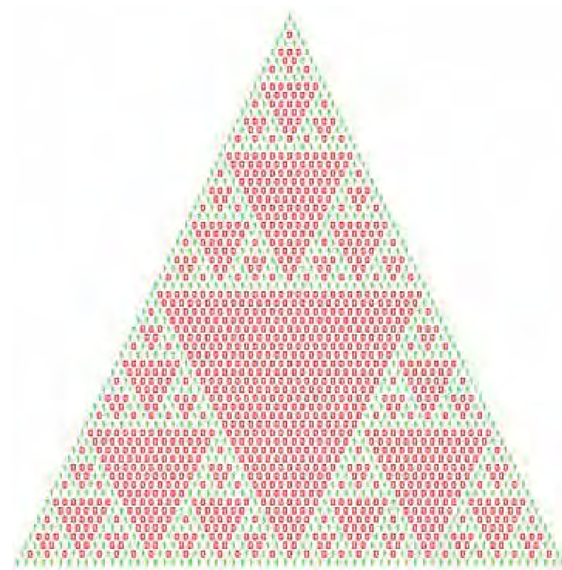


Figure 14. The binary Pascal's Triangle is also the Sierpinski Triangle

Of course, you can guess how to obtain the Sierpinski Triangle by iteration. Begin with a triangle, remove the middle triangle in step 1, which will leave behind three triangles to which you do the same! And just repeat this process forever.

The fact that Pascal's triangle is symmetric upon reflection, led to the design below (Figure 15)—made up of row over row of mirror-symmetric designs for the word "Pascal" increasing in size as we go down the rows. We call this design "*a* Pascals Triangle" (a triangle *made up of many* "Pascals") as opposed to "*the* Pascal's triangle" (the triangle *of or belonging to* Pascal). (Author's note: This design was created under psignificant work pressure. Can you guess why?)

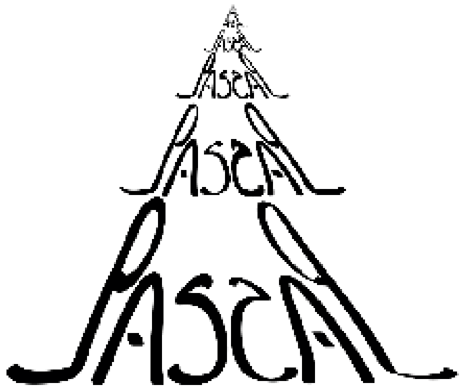


Figure 15. An ambigrammist's approach to Pascals Triangle (as opposed to Pascal's Triangle). What a difference an apostrophe makes!

Figure 16. An ambigram for "Hidden Beauty", whose beauty is not hidden from anyone!

In conclusion

We have explored many ideas in this article—self-similarity, iteration, recursion, infinity, and attempted to represent them graphically even while connecting them to deeper mathematical ideas. We hope that like us, you too experienced many feelings when you encountered these ideas—feelings of wonder, amusement, surprise, or the indescribable feeling when one finds something beautiful. We hope that these feelings make you wish to create something new, and perhaps dream up interesting things to share with your friends. As Borges eloquently said, "The mind was dreaming. The world was its dream." There is a lot of beauty one can find, hidden away in the world of ideas. We close with an ambigram for "Hidden Beauty" in Figure 16, where the word *hidden* becomes *beauty* when rotated 180 degrees!

Answer to the Puzzle in Figure 7: The "B" in "Benoit B Mandelbrot" stands for Benoit B Mandelbrot... and so on forever! Here is another way of representing the same idea, that we call, "Just let me B."

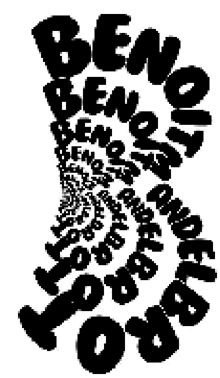


Figure 17. "Just let me B:" A fractal design for Benoit B Mandelbrot designed to answer the question "what does the B in Benoit B Mandelbrot stand for?"

PUNYA MISHRA, when not creating ambigrams, is professor of educational technology at Michigan State University. GAURAV BHATNAGAR, when not teaching or doing mathematics, is Senior Vice-President at Educomp Solutions Ltd. They have known each other since they were students in high-school.

Over the years, they have shared their love of art, mathematics, bad jokes, puns, nonsense verse and other forms of deep-play with all and sundry. Their talents however, have never truly been appreciated by their family and friends.

Each of the ambigrams presented in this article is an original design created by Punya with mathematical input from Gaurav. Please contact Punya if you want to use any of these designs in your own work.

To you, dear reader, we have a simple request. Do share your thoughts, comments, math poems, or any bad jokes you have made with the authors. Punya can be reached at punya@msu.edu or through his website at http://punyamishra.com and Gaurav can be reached at bhatnagarg@gmail.com and his website at http://gbhatnagar.com/.

Of Art and Mathematics

Paradoxes: True AND/OR False?

Part 1 of 2

Punya Mishra & Gaurav Bhatnagar

This is the first sentence of this article.

Clearly the sentence above is true (not highly informative but true). Contrast this to the next sentence, below:

This is the first sentence of this article.

Now the second statement, though identical to the first, is clearly false.

Such sentences that speak about themselves are called *self-referential* sentences, because they are, in a way, looking at themselves in the mirror and describing themselves. Figure 1, is a design for the word "reference" so it looks the same when reflected in a mirror.

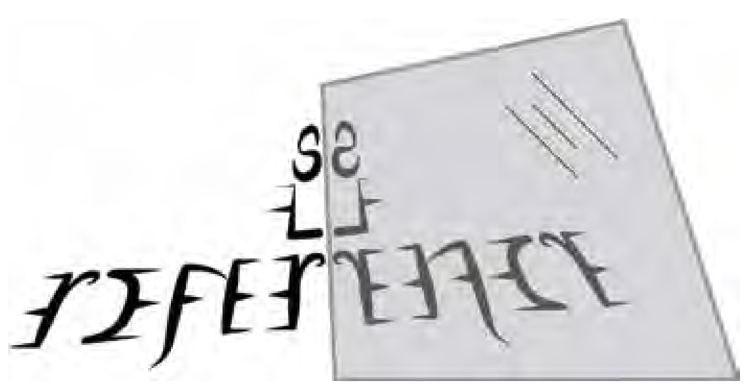


Figure 1. Self-reference looks in a mirror. The word "self-reference" is written in a manner that it looks the same when reflected in a mirror (a wall reflection).

Keywords: *Truth value, self-reference, paradox, axiom, theorem, consistency, circular argument, proof, Zeno, ourobouros, ambigram*

Figure 2. An ambigram for Paradox, the subject of this column

Such self-referential sentences sometimes lead to paradoxes, and paradoxes are the topic of this article. As usual we use the medium of ambigrams to communicate some of these paradoxical ideas (see Figure 2 for an ambigram of Paradox). And we produce some graphical paradoxes of our own for you to think about.

Mathematical Truth

To understand what self-referential statements have to do with mathematics we need to get a bit deeper into what mathematicians mean by the words *true* and *false*. A mathematical theory consists of a large number of statements. There are two special types of true statements in any mathematical theory–axioms and theorems.

For example, consider the development of plane geometry. We begin with certain axioms (such as: given a line and a point not on the line, there is exactly one line through that point parallel to the given line). Axioms are all considered to be true. Now by following the rules of logic, from Axioms one *proves* some other statements that are called theorems. If the proof is valid, we say the theorem is true. For example, a theorem is: The sum of three angles of a triangle is equal to two right angles. Each theorem is proved using the axioms, or the previously proved theorems. Figure 3 includes an ambigram of the word "axiom" that is then used over and over again to create an ambigram of the word "theorem."

Each statement in this theory is either true or false—it cannot be both, otherwise there will be a contradiction. And we will see shortly that contradictions are not allowed in mathematics.

Figure 3. Rotational ambigrams for the words "axiom" and "theorem"' – except that the word "theorem" is both an ambigram and constructed from the multiple axioms

Puzzle:

Can you decipher these strange squiggles below? Hint: There are two words related to this article

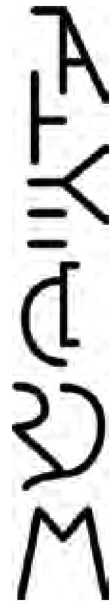


Figure 4. What do these squiggles mean?

In this theory, the axioms are taken to be true. However it is not necessary that the axioms are 'true' in every context. For example, the axioms of plane geometry are true in the idealized plane, but do not hold for the surface of the sphere, where 'lines' are simply *great circles*, which are formed by the intersection of the sphere with a plane passing through the center of the sphere. The equator, and lines of longitude are examples of great circles on a spherical globe. In this geometry, there is no line parallel to the given line from a point not on the line! This is because two great circles always meet. But surely the geometry of the sphere is equally "true" in the real world. (This kind of geometry, on the surface of the sphere, is called Riemannian Geometry).

What mathematical theories try to achieve is a consistency, where by consistency we mean: given the axioms and theorems proved within the theory (using the rules of logic), none of the statements contradict each other. Proofs are means to convince ourselves that the statements are "true" in the mathematical theory.

In developing a mathematical theory, one needs to be careful to avoid a circular proof. A circular proof is when the proof of a statement uses the statement itself! Figure 5 is a reflection chain ambigram of the word "proof" — a visual circular proof!

A circular argument can be difficult to find. Say in proving a statement P we use the truth of a

Figure 5. A visual representation of a circular proof! This design reads the same both at the front (as in red) or at the back — or even when read in a mirror.

statement Q. But the proof of the statement Q involves the statement P. A good example of circular reasoning is in the book Catch 22,

> "You mean there's a catch?"
>
> "Sure there's a catch", Doc Daneeka replied. "Catch-22. Anyone who wants to get out of combat duty isn't really crazy."
>
> There was only one catch and that was Catch-22, which specified that a concern for one's own safety in the face of dangers that were real and immediate was the process of a rational mind. Orr was crazy and could be grounded. All he had to do was ask; and as soon as he did, he would no longer be crazy and would have to fly more missions. Orr would be crazy to fly more missions and sane if he didn't, but if he was sane, he had to fly them. If he flew them, he was crazy and didn't have to; but if he didn't want to, he was sane and had to. Yossarian was moved very deeply by the absolute simplicity of this clause of Catch-22 and let out a respectful whistle.
>
> "That's some catch, that Catch-22," he observed.
>
> "It's the best there is," Doc Daneeka agreed.

Or in the character Tippler in the Little Prince who says he drinks so that he may forget that he is ashamed of drinking! As the little prince says,

> "The grown-ups are certainly very, very odd."

In mathematics circular proofs show up when something that is assumed is then used to prove the same thing. For instance here is a circular proof of the Pythagorean theorem.

> Let ΔABC be a right triangle with sides a, b, c. As usual, let c be the hypotenuse, the side opposite the right angle C. We know that $\sin B = b/c$ and $\cos B = a/c$.
>
> Now using the elementary trigonometric identity $\cos^2 B + \sin^2 B = 1$, we find that $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, or $a^2 + b^2 = c^2$, as required.

The only problem with this proof is that it presupposes the Pythagorean theorem—the very theorem that it sets out to establish.The proof of $\cos^2 B + \sin^2 B = 1$ relies on the Pythagorean Theorem! This is a good example of a vicious circle (see the design for *ouroboros*, Figure 6, for another, more lethal, variant of a vicious circle!).

Why do Mathematicians not allow any contradictions in the theories they build?

Mathematicians avoid contradictions because they can completely destroy the entire theory. This is because of a theorem of logic: *a false proposition implies any proposition*. Given that a false statement implies *any* statement, there is not much point in having a theory that has false

Figure 6. A chain rotation ambigram for the word "ouroboros" representing the idea of a snake eating its own tail. The idea of the ouroboros has recurred throughout history – such as the image in the middle, which is from a late medieval alchemical manuscript (courtesy Wikimedia Commons).

Figure 7. An ambigram about the relationship of math to truth

statements. For instance, on the one hand, we can prove a statement such as: There are an infinite number of prime numbers (as Euclid did over 2000 years ago). However, if even *one* false statement creeps into our mathematical universe, we can also prove that: There are only finitely many prime numbers! Or that there are exactly 317 prime numbers. Or that there are no prime numbers! Or that prime numbers are made of sweet buttermilk!

An example of a 'Proof' using a false proposition is this famous (probably apocryphal)story about the philosopher and mathematician Bertrand Russell (as retold by Raymond Smullyan in his classic book *What is the name of this book?).* Russell once told a dinner audience that "a false proposition implies any proposition." He was challenged to show that if $2 + 2 = 5$ (clearly a false statement) then he could prove that he (Russell) is the Pope. Russell then responded as follows:

> Given that $2 + 2 = 5$. Subtract 3 from both sides to get $1 = 2$. Now consider the following statements: The Pope and I are two. But $2 = 1$. So the Pope and I are one. Thus I am the Pope!

Note that starting from a false statement we end up with a nonsensical statement that "Russell is the Pope'. Thus something is wrong with the argument.

Mathematicians avoid contradictions like the plague (even more than writers avoid clichés). This is the reason why we insist on proofs in mathematics—to convince ourselves that all the statements are true. Figure 6 is a design where "math" rotates to read the word "truth."

Sometimes contradictions lead to paradoxes (or apparent paradoxes). Paradoxes are contradictory statements and have to be false. But since false statements are not allowed, there has to be some flaw in the reasoning. Resolving these paradoxes helps us understand the flaws in our reasoning. And more importantly, thinking about these paradoxical situations is fun.

Before we get into some serious self-contradictory paradoxes here is one that goes back a while – and one that turns out not to be a paradox if addressed with the right mathematical tools.

Zeno's Paradox

Zeno's paradoxes are about the impossibility of motion. A simple example is as follows. Suppose you have to go from a point A to a point B, which are 1 km distant from each other. Then first you have to reach halfway, a distance of half a km away. Then you have to go from mid point of AB (say A_1) to B. Again you have to first go half the distance A_1B which is one-fourth of a km. Next we have to go half the remaining distance, that is, one-eighth of a km. Going on in this fashion, Zeno asserted that we can never reach B. In other words, it is impossible to go from A to B. Thus Zeno showed by this argument that motion is impossible!

What is wrong with Zeno's argument? Zeno's paradoxes forced philosophers and mathematicians to think of the *continuum* and concepts such as infinite series. In our example above, we find that

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

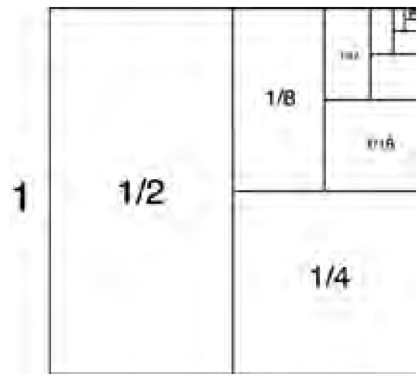


Figure 8. A 'proof by pictures' of the sum of the geometric series and how an infinite number of additions can lead to a finite sum

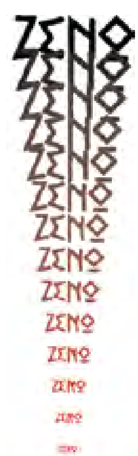


Figure 9. A visual Zeno Paradox, where "Zeno" gradually transforms to "Zero" – where the letter "n" changes step by step to the letter "r." Is Zeno ever Zero?

which follows from the formula for the sum of the geometric series. Figure 8 shows a "proof by pictures" of this series. We can use the concept of infinite series to resolve Zeno's paradox, by noting that the sum of an infinite number of additions can be a finite quantity.

Figure 9 shows an ambigrammatic approach to Zeno's paradox; here the word Zeno tends to Zero!

In the Geometric Series, the infinite sum is a finite quantity. The ambigram of Figure 10 is about the word "Finite" written in such a manner that it becomes the symbol for infinity!

Figure 10. Finite reflection in a circle. The word finite repeats in a circle – and is also reflected in a mirror. Taken together the main image and its reflection from the symbol for infinity.

In conclusion

With this we come to the end of our first part of our reconnaissance of the domain of paradoxes in mathematics. There is a lot more to come…but for that you will have to wait for part 2 of this article.

So with that, we should let you know that though it may seem that way, *this* sentence is surely not the last word on the topic. This is. No. This. Word.

Answer to puzzle: If you place a mirror vertically along the middle of the squiggles you will see two words – Axiom and Theorem (as follows).

Figure 11. Solution to Puzzle 1

PUNYA MISHRA, when not pondering visual paradoxes, is professor of educational technology at Michigan State University. **GAURAV BHATNAGAR**, when not reflecting on his own self, is Senior Vice-President at Educomp Solutions Ltd. They have known each other since they were students in high-school.

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To you, dear reader, we have a simple request. Do share your thoughts, comments, math poems, or any bad jokes you have made with the authors. Punya can be reached at punya@msu.edu or through his website at http://punyamishra.com and Gaurav can be reached at bhatnagarg@gmail.com and his website at http://gbhatnagar.com/.

Of Art and Mathematics

Paradoxes:

Part 2 of 2

feature

PUNYA MISHRA & GAURAV BHATNAGAR

This is not the first sentence of this article.

The above sentence can be both true and false. It is clearly the first sentence of *this* article. So it is false, because it says it is not the first sentence! But because this is part 2 of our article on Paradoxes, if we regard both parts as one article, it is true! We leave it to you to resolve this paradox.

In the first part of this two-part exposition on paradoxes in mathematics, we introduced the idea of self-reference, the nature of mathematical truth, the problems with circular proofs and explored Zeno's Paradox. In this part we delve deeper into the challenges of determining the 'truth value' of pathological self-referential statements, visual paradoxes and more.

Self - Reference and Russell's Paradox

There is a class of paradoxes that arise from objects referring to themselves. The classic example is Epimenides Paradox (also called the Liar Paradox). Epimenides was a Cretan, who famously remarked "All Cretans are liars." So did Epimenides tell the truth? If he did, then he must be a liar, since he is a Cretan, and so he must be lying! If he was lying, then again it is not the case that all Cretans are liars, and so

Keywords: *Paradox, Circular proof, Zeno's paradoxes, Russell's paradox, Epimenides liar paradox, Self-reference, Contradiction, Escher, Penrose, Jourdain's paradox, Triangle, Necker's Cube*

Figure 1. An ambiguous design that can be read as both "true" and "false."

he must be telling the truth, and that cannot be! Figure 1 is an ambiguous design that can be read as both "true" and "false."

The artwork of M.C. Escher (such as his famous illustration that shows two hands painting each other) provides many visual examples of such phenomena. Another older analogy or picture is that of the *ouroboros*—an image of a snake eating its own tail (how's *that* for a vicious circle!). An ambigram of *ouroboros* was featured in our first article on paradoxes.

Here is another variation of the Liar Paradox. Consider the following two sentences that differ by just one word.

This sentence is true.

This sentence is false.

The first is somewhat inconsequential – apart from the apparent novelty of a sentence speaking to its own truth value.

The second, however, is pathological. The truth and falsity of such pathologically self-referential statements is hard to pin down. Trying to assign a truth value to it leads to a contradiction, just like in the Liar Paradox. Figure 2 is a rotational ambigram that reads "true" one way and "false" when rotated 180 degrees.

Figure 2: Rotational ambigram that reads "False" one way and "True" the other. (This design was inspired by a design by John Longdon.)

A variant of this (that does not employ self-reference) is also known as the Card paradox or Jourdain's paradox (named after the person who developed it). In this version, there is a card with statements printed on both sides. The front says, "The statement on the other side of this card is TRUE," while the back says, "The statement on the other side of this card is FALSE." Think through it, and you will find that trying to assign a truth value to either of them leads to a paradox!

Figure 3 combines the liar's paradox and Jourdain's paradox (in its new ambigram one-sided version) into one design.

Figure 3: Two paradoxes in one. Inside the circle is the ambigram for the pair of sentences "This sentence is True/This sentence is False". The outer circle is an original design for the Jourdain two-sided-card paradox, which can, due to the magic of ambigrams, be reduced to being printed on just one side!

Another interesting example is the sentence: "This sentence has two ewrrors." Does this indeed have two errors? Is the error in counting errors itself an error? If that is the case, then does it have two errors or just one?

What is intriguing about the examples above is that they somehow arise because the sentences refer to themselves. The paradox was summarized in the mathematical context by Russell, and has come to be known as Russell's paradox. Russell's paradox concerns sets. Consider a set R of all sets that do not contain themselves. Then Russell asked, does this set R contain itself? If it does contain itself, then it is not a member of R. But if it is not a member of R, then it does contain itself.

Russell's Paradox was resolved by banning such sets from mathematics. Recall that one thinks of a set as a *well-defined* collection of objects. Here by well-defined we mean that given an element a and a set A, we should be able to determine whether a belongs to A or not. So Russell's paradox shows that a set of all sets that do not contain themselves is not well-defined. By creating a distinction between an element and a set, such situations do not arise. You could have sets whose members are other sets, but an element of a set cannot be the set itself. Thus, in some sense, self-reference is not allowed in Set Theory!

Visual contradictions

Next, we turn to graphic contradictions, where we use ambigrams to create paradoxical representations.

Figure 4: A somewhat inelegant design that captures a visual paradox – the word "asymmetry" written in a symmetric manner.

Figure 4 shows an ambigram for asymmetry, but it is symmetric. So in some sense, this design is a *visual contradiction*! But it is not a very elegant solution – which in some strange way is appropriate.

Recall the idea of self-similarity from our earlier column, where a part of a figure is similar to (or a scaled-down version of) the original. Here is an ambigram for similarity which is made up of small pieces of self (Figure 5). Should we consider this to be self-similarity?

Another set of visual paradoxes have to do with the problems that arise when one attempts to represent a world of 3 dimensions in 2 dimensions – such as in a painting or drawing. The Dutch artist M.C. Escher was the master at this. His amazing paintings often explore the paradoxes and impossible figures that can be created through painting. For instance, he took the mathematician and physicist Roger Penrose's image of an impossible triangle and based some of his work on it (Figure 6).

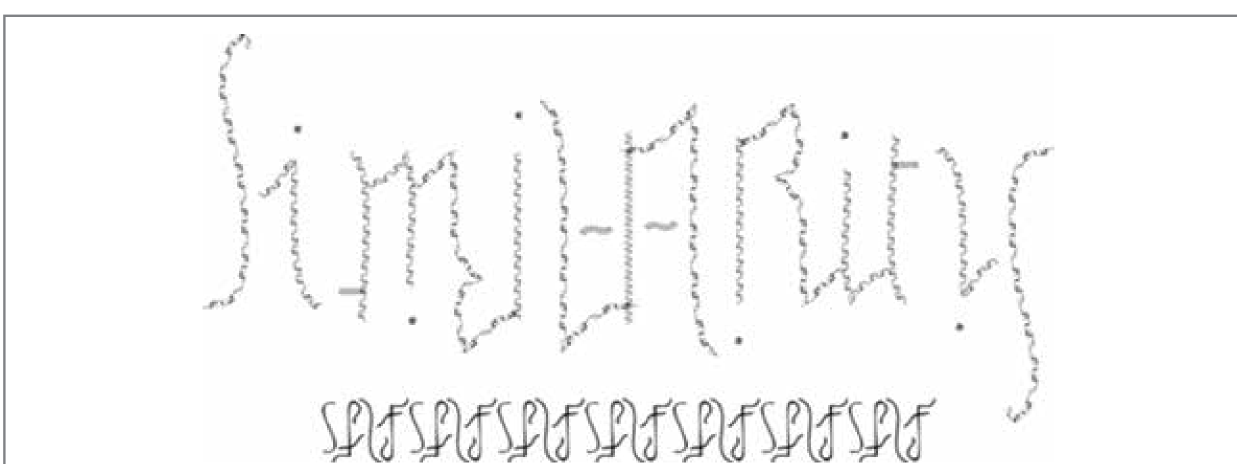

Figure 5: Here is an ambigram for Similarity which is made up of small pieces of Self. Should we consider this to be self-similarity?

Figure 6. A Penrose Triangle – a visual representation of an object that cannot exist in the real world.

Puzzle: What is the relationship between a Penrose Triangle and a Möbius strip?

As homage to M.C. Escher, we present below (Figure 7) a rotational ambigram of his name written using an impossible font!

Figure 7: Rotationally symmetric ambigram for M.C. Escher written using an impossible alphabet style.

As it turns out, the Penrose Triangle is also connected to another famous geometrical shape, the Möbius strip. A Möbius strip has many interesting properties, one of which is that it has only one side and one edge (Figure 8).

Figure 8. An unending reading of the word Möbius irrespective of how you are holding the paper!

Another famous impossible object is the "impossible cube." The impossible cube builds on the manner in which simple line drawings of 3D shapes can be quite ambiguous. For instance, see the wire-frame cube below (also known as the Necker Cube). This image usually oscillates between two different orientations. For instance, in Figure 9, is the person shown sitting *on* the cube or magically stuck to the ceiling *inside* it?

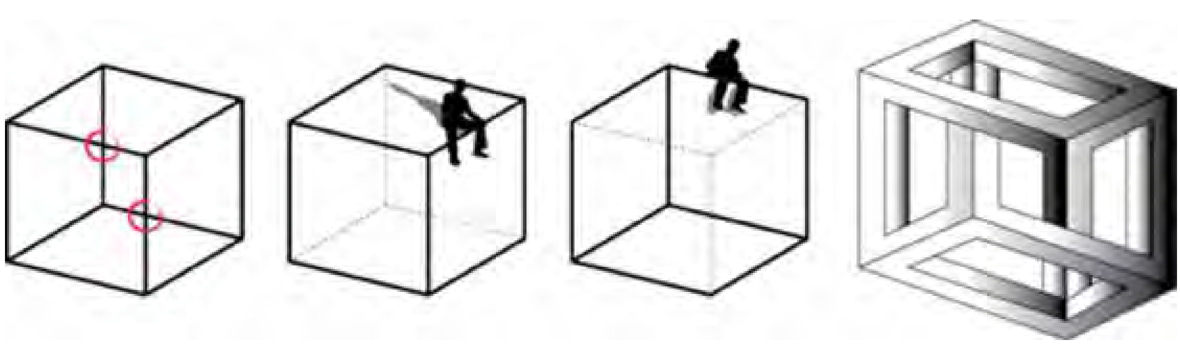

Figure 9. The Necker Cube – and how it can lead to two different 3D interpretations and through that to an impossible or paradoxical object.

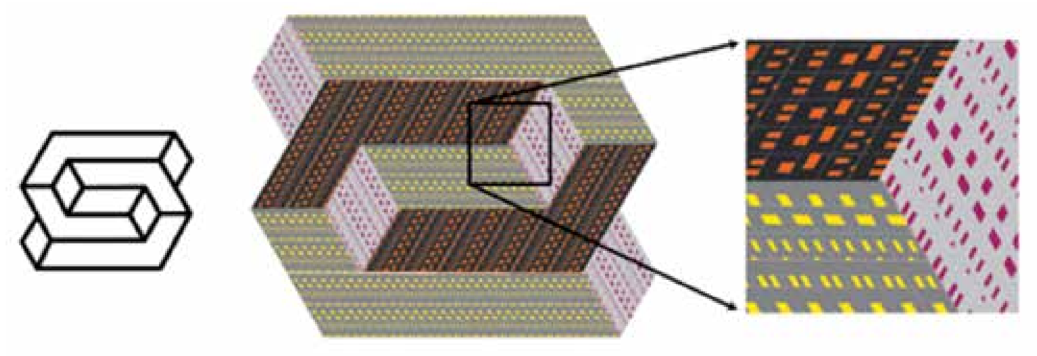

Figure 10: The impossible cube? In this design the word "cube" is used to create a series of shapes that oscillate between one reading and the other.

Figure 11: An impossible typeface based on the Necker Cube and Penrose Triangle. Spelling the word "Illusions."

These images oscillate between two opposite incommensurable interpretations, somewhat like the liar paradoxes we had described earlier. Figure 10 is another ambiguous shape that can be read two ways! What is cool about that design is that each of these shapes is built from tiny squares that read the word "cube."

These representations fool our minds to see things in ways that are strange or impossible. These are visual paradoxes, or illusions, as reflected in the design in Figure 11, which is the word "illusions" represented using an impossible font (akin to the Penrose Triangle or Necker Cube).

Mathematical Truth and the Real World

One of the most fundamental puzzles of the philosophy of mathematics has to do with the fact that though mathematical truths appear to have a compelling inevitability (from axiom to theorem via proof) and find great applicability in the world, there is little we know of why this is the case. The physicist Wigner called it the "unreasonable effectiveness of mathematics" to explain, understand and predict the phenomena in the real world. The question is how something that exists in some kind of an "ideal" world can connect to and make sense in the "real" world we live in.

Figure 12: The Ideal-Real ambigram, representing the paradoxical thought that the Real world often appears to be a reflection of the Ideal mathematical theory!

Figure 12 maps the word "ideal" to "real." Is the ideal real – and real just a mere reflection of the ideal? Or vice versa?

Clearly this is not an issue that will be resolved anytime soon – but it is intriguing to think about.

So with that, we bid adieu, but before we depart we would like to bring you the following self-serving public announcement.

This is the last sentence of the article. No this is. This.

Answer to Puzzle:

The Möbius Strip and the Penrose Triangle have an interesting relationship to each other. If you trace a line around the Penrose Triangle, you will get a 3-loop Möbius strip. M.C. Escher used this property in some of his most famous etchings.

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Of Art & Math

Punya Mishra & Gaurav Bhatnagar

Introduction • Symmetry • Self-Similarity • Paradoxes

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