in the classroom

Of Art & Math:

Introducing Symmetry

Punya Mishra
Gaurav Bhatnagar

In our November column we introduced the concept of ambigrams—the art of writing words in surprisingly symmetrical ways. Consider an ambigram of the word "Symmetry" (Figure 1).

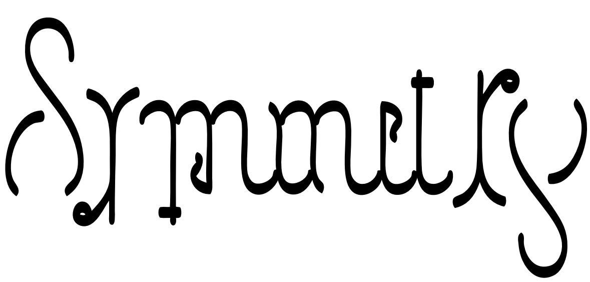


Figure 1. A symmetric ambigram for Symmetry

The design itself displays *rotational symmetry, i.e. it looks the same even when rotated by 180 degrees.* In other words, it remains *invariant* on rotation. Figure 2 shows an ambigram for "invariant" with a similar property.

Keywords: *Ambigrams, calligraphy, symmetry, perception, mapping, transformation, reflection, translation, invariance, function, inverse*

Figure 2. An ambigram for "invariant" that remains invariant on rotation

Invariance can also be seen in reflection. Figure 3 gives a design for the word "algebra" that is invariant upon reflection, but with a twist. You will notice that the left hand side is NOT the same as the right hand side and yet the word is still readable when reflected. So the invariance occurs at the level of meaning even though the design is not visually symmetric!

Figure 3. An ambigram for 'algebra' that remains invariant on reflection. But is it really symmetric?

In this column, we use ambigrams to demonstrate (and play with) mathematical ideas relating to symmetry and invariance.

There are two common ways one encounters symmetry in mathematics. The first is related to graphs of equations in the coordinate plane, while the other is related to symmetries of geometrical objects, arising out of the Euclidean idea of congruence. Let's take each in turn.

Symmetries of a Graph

First let us examine the notions of symmetry related to graphs of equations and functions. All equations in x and y represent a relationship between the two variables, which can be plotted on a plane. A graph of an equation is a set of points (x, y) which satisfy the equation. For example, $x^2 + y^2 = 1$ represents the set of points at a distance 1 from the origin—i.e. it represents a circle.

Let (a, b) be a point in the first quadrant. Notice that the point $(-a, b)$ is the reflection of (a, b) in the y-axis (See Figure 4a). Thus if a curve has the property that $(-x, y)$ lies on the curve whenever (x, y) does, it is symmetric across the y-axis. Such functions are known as even functions, probably because $y = x^2, y = x^4, y = x^6, \ldots$ all have this property. Figure 4b shows the graph of $y = x^2$; this is an even function whose graph is a parabola.

Similarly, a curve is symmetrical across the origin if it has the property that $(-x, -y)$ lies on the curve whenever (x, y) does. Functions whose graph is of this kind are called odd functions, perhaps because $y = x, y = x^3, y = x^5, \ldots$ all have this property. See Figure 4c for an odd function.

A graph can also be symmetric across the x-axis. Here $(x, -y)$ lies on the curve whenever (x, y) does. The graph of the equation $x = y^2$ (another parabola) is an example of such a graph. Can a (real) function be symmetric across the x-axis?

Figure 5 shows a chain ambigram for "parabola". Compare the shape of this ambigram with the graph in Figure 4b. The chain extends indefinitely, just like the graph of the underlying equation!

Figure 5. A parabolic chain ambigram for "parabola"

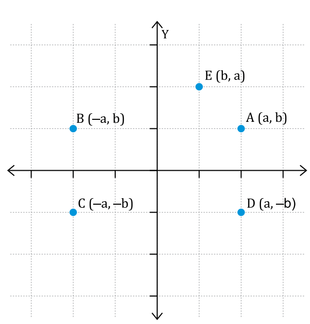


Figure 4a. The point *A (a, b)* and some symmetric points

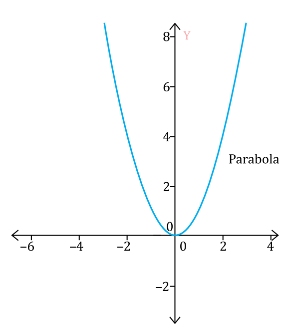


Figure 4b. An even function: $y = x^2$

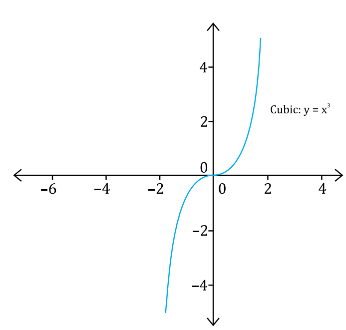


Figure 4c. An odd function: $y = x^3$

The ambigram for "axis of symmetry" (Figure 6) is symmetric across the y-axis. You can see a red y as a part of the x in the middle. So this ambigram displays symmetry across the y axis. At the same time it is symmetric across the letter x!

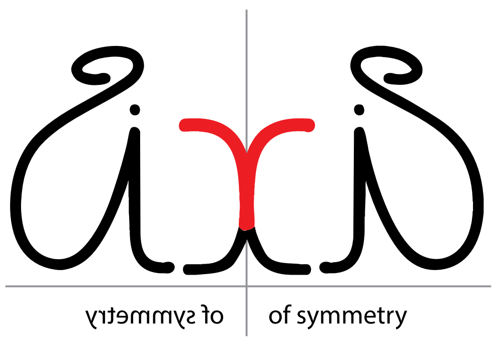


Figure 6. The axis of symmetry: Is it the y-axis or the x?

Another possibility is to interchange the x and y in an equation. Suppose the original curve is C_1 and the one with x and y interchanged is C_2. Thus if (x, y) is a point on C_1, then (y, x) lies on C_2. By looking at Figure 4a, convince yourself that the point (b, a) is the reflection of (a, b) in the line $y = x$, the straight line passing through the origin, and inclined at an angle of 45° to the positive side of the x-axis. Thus the curve C_2 is obtained from C_1 by reflection across the line $y = x$. If C_1 and C_2 (as above) are both graphs of functions, then they are called inverse functions. An example of such a pair: the *exp* (exponential $y = e^x$) and *log* (logarithmic $y = \ln x$) functions (Figure 7).

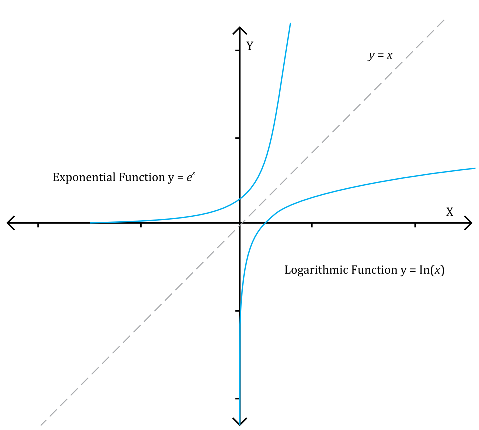


Figure 7. Inverse functions are symmetric across the line $y=x$.

Figure 8 is a remarkable design that where *exp* becomes *log* when reflected in the line $y = x$.

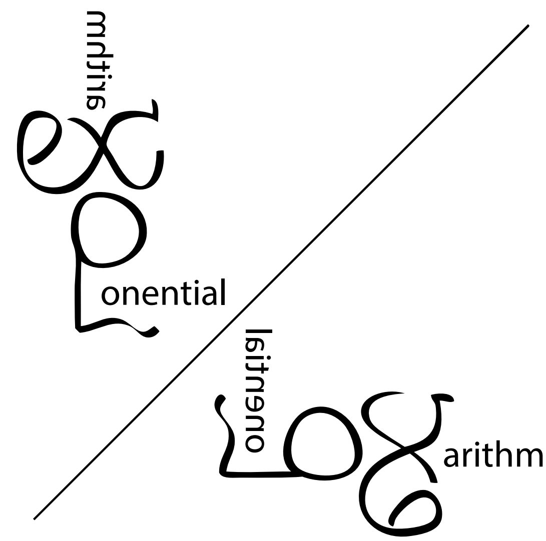


Figure 8: *Exp* becomes *Log* when reflected across the diagonal line!

A great example of an inverse function is the hyperbola $y = 1 / x$, defined for all non-zero real numbers x (Figure 9). Its inverse is obtained by interchanging x with y. But $x = 1 / y$ can be written $y = 1 / x$. So it is its own inverse, and thus symmetric across the line $y = x$.

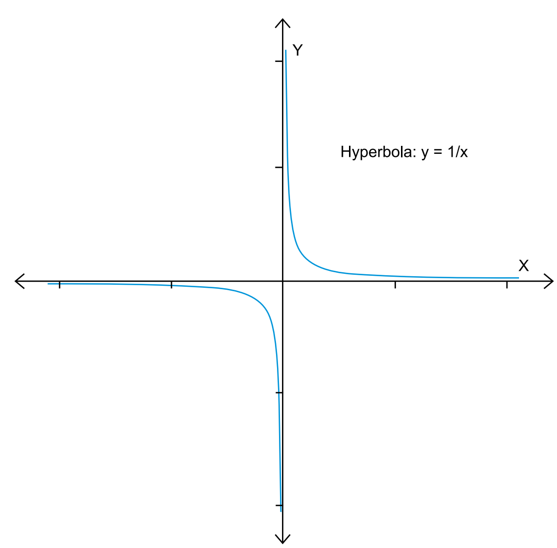


Figure 9. The symmetrical graph of the hyperbola. It is its own inverse. And it's odd, too!

The ambigram for "inverse" in Figure 10 is inspired by the hyperbola.

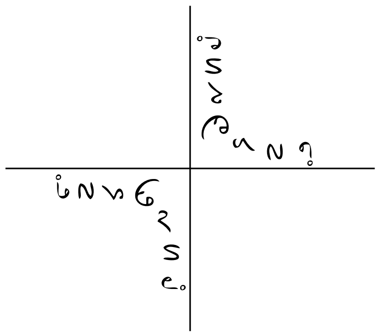


Figure 10. An ambigram for the word "inverse" shaped like a hyperbola.

It is symmetric across the origin and across the line joining the two S's.

Seeking congruence

Another type of symmetry consideration arises from the notion of congruence in plane geometry. Two objects are considered to be congruent if one object can be superposed on the other through rotation, reflection and/or translation.

Figure 11 shows an ambigram of the word "rotate".

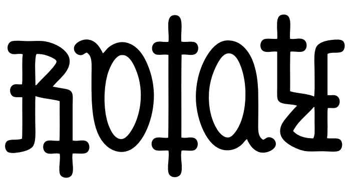


Figure 11. An ambigram for "rotate". What happens when you rotate it through 180°?

This leads to the question: if we can rotate "rotate" can we reflect "reflect"? Figure 12 is an ambigram for "reflect" that is symmetric around the vertical line in the middle.

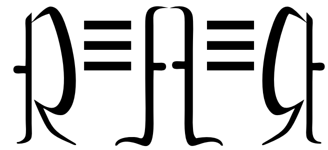


Figure 12. An ambigram for "reflect." What happens when you hold it to a mirror?

Finally, the third operation is translation. An example of this symmetry is shown by the chain ambigram for sine in Figure 13.

Figure 13. A sine wave ambigram. It displays translation symmetry.

The sine function satisfies many symmetry properties. Perhaps the most important of them is that it is periodic, i.e., if you shift (in other words, translate) the functions by 2π, then you get the same function back. In addition, it is an odd function, and the ambigram is both periodic and odd.

One can of course combine these transformations. This is best understood by looking at the symmetries of an equilateral triangle (see Figure 14) involving both rotation and reflection.

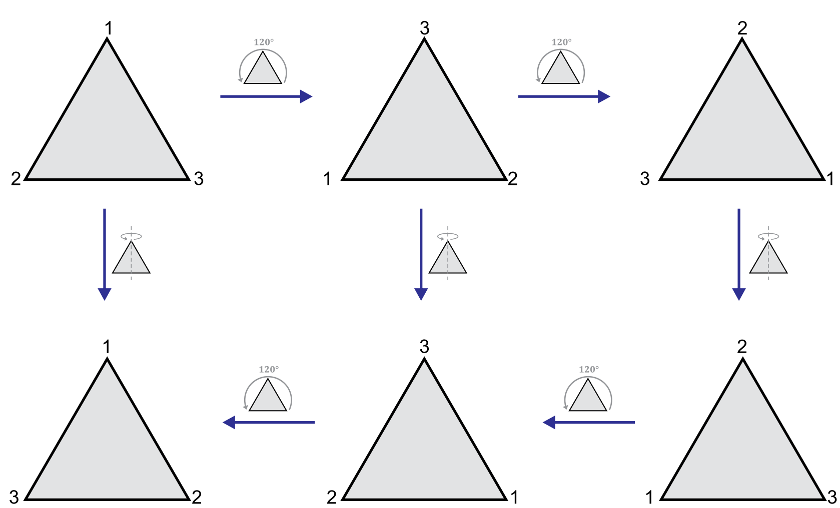


Figure 14. The 6 symmetries of an equilateral triangle.

The 6 symmetries of the equilateral triangle are all found from two fundamental operations: (a) rotation by 120°; and (b) reflection across the line passing through the vertex 1, and At Right Angles to the base of the triangle. Figure 15 shows ambigrams for the word triangle and pentagon. Do they display all the symmetries of the equilateral triangle and regular pentagon (respectively)?

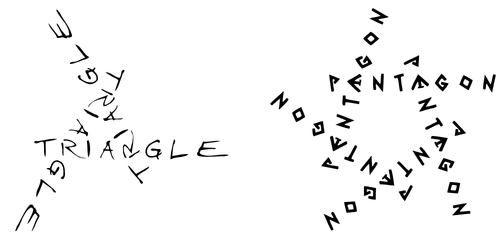


Figure 15: Ambigrams for "triangle" and "pentagon" showing rotational symmetries.

In conclusion

Clearly we have just scratched the surface of the power of symmetry as an idea in mathematics. The philosopher Aristotle once observed that, "the mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful." We agree with Aristotle, but perhaps we would have said "arts" instead of "sciences." In our next article, we will continue to use ambigrams to explore more beautiful mathematical ideas.

Our last article had a secret message. The first letter of each paragraph read: "Martin Gardner lives on in the games we play". This is our homage to Martin Gardner whose writings inspired us when we were growing up. This article has a different puzzle (see Figure 16).

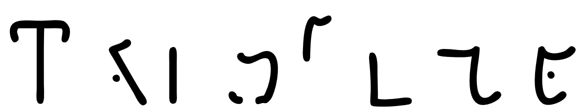
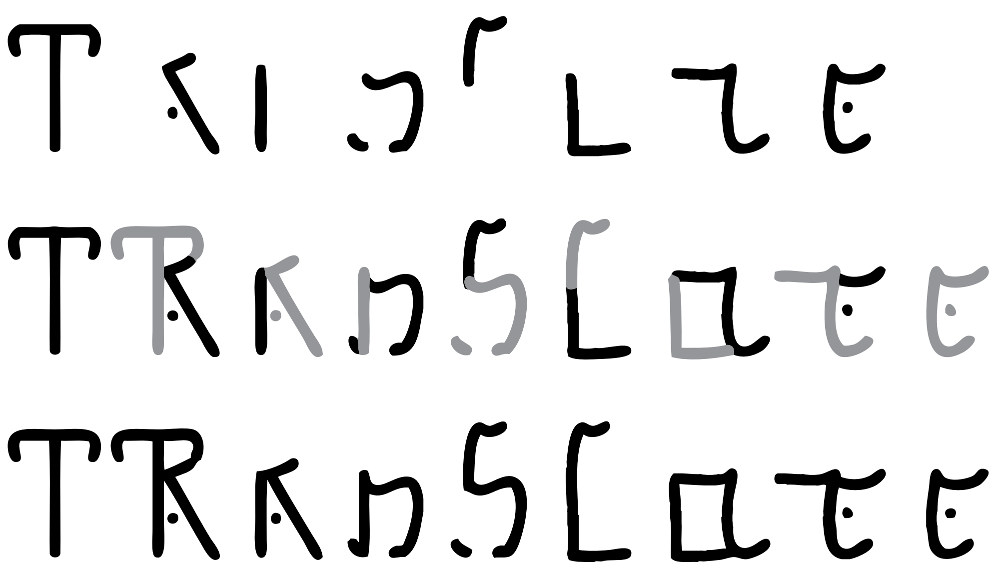


Figure 16. Can you translate this hieroglyphic code? The answer appears below.

ANSWER TO THE PUZZLE

We asked you to "translate" the code. Once you translate (i.e. move) the shapes and align them, you get the answer—the word "Translate." An example of translation symmetry!



About the authors

PUNYA MISHRA, when not creating ambigrams, is professor of educational technology at Michigan State University. GAURAV BHATNAGAR, when not teaching or doing mathematics, is Senior Vice-President at Educomp Solutions Ltd.

Loving both math and art, Punya's and Gaurav's collaboration began over 30 years ago when they were students in high-school. Since then, they have individually or collectively, subjected their friends, family, classmates, and students to a never ending stream of bad jokes, puns, nonsense verse and other forms of deep play. To their eternal puzzlement, their talents have not always been appreciated by their teachers (or other authority figures). Punya's email address is punya@msu.edu and his website is at http://punyamishra.com. Gaurav's email address is bhatnagarg@gmail.com and his website is at http://gbhatnagar.com

All the ambigrams presented in this article are original designs created by Punya Mishra (unless otherwise specified). Please contact him if you need to use them in your own work.

You, dear reader, are invited to share your thoughts, comments, math poems, or original ambigrams at the addresses above.

A RIVER PUZZLE FROM MASTER PUZZLIST SAM LOYD

Two ferry boats ply the same route between ports on opposite sides of a river. They set out simultaneously from opposite ports, but one is faster than the other, so they meet at a point 720 yards from the nearest shore. When each boat reaches its destination, it waits 10 minutes to change passengers, then begins its return trip. Now the boats meet at a point 400 yards from the other shore. How wide is the river?

Comment from Sam Loyd: "The problem shows how the average person, who follows the cut-and-dried rules of mathematics, will be puzzled by a simple problem that requires only a slight knowledge of elementary arithmetic. It can be explained to a child, yet I hazard the opinion that ninety-nine out of every hundred of our shrewdest businessmen would fail to solve it in a week. So much for learning mathematics by rule instead of common sense which teaches the reason why!"

See ***page* 53** for the solution.

Source: http://www.futilitycloset.com/category/puzzles/